THE ROUTH FUNCTION OF A SLENDER PLASMA FILAMENT

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One of the possible ways to study the stability of a plasma conductor with a current in an external magnetic field is to consider the conductor as an electromechanical system with an infinite number of degrees of freedom. If it is possible to find in explicit form the Lagrange function describing the motion of the conductor near the equilibrium position, then the stability study can be made using known techniques. Specifically, for slender plasma filaments experiencing longwave disturbances of the zigzag and sausage type it is possible to find the Lagrange function for very general assumptions on the field geometry. This makes it possible to study the plasma stability with respect to the disturbances which are most hazardous from the MHD viewpoint in many devices with complex magnetic fields in which the plasma has the form of a slender straight or closed filament.

This method was first used by Levin and Rabinovich [1] to study dynamic stabilization of a plasma filament by a high-frequency quadrupole magnetic field, suggested by Osovets [2]. Some other quadrupole dynamic stabilization versions were then studied [3] using the approach developed in [1]. In this paper this method is extended to the case of magnetic fields of arbitrary geometry. In section 1 we prove a theorem in accordance with which an ideal (in the sense of no dissipation) electromechanical system with closed currents in the quasi-stationary approximation is described by the Routh function, which is the difference between the mechanical lagrangian of the system and the magnetic energy of the currents in the self-magnetic field. With respect to the mechanical variables this function plays the role of the conventional Lagrange function. In section 2 we find in general form the expression for the self-magnetic energy of a slender closed plasma conductor experiencing smooth (longwave) disturbances of the zigzag sausage type. The expression for the mechanical Lagrangian in the case of a circular ring was obtained in [1].

1. As is known (see, for example, [4]) combining of the electrical and mechanical equations of motion of moving conductors with currents into a common dynamic system is accomplished by simple addition of the mechanical L_M and electromagnetic L_E Lagrange functions

$$L = L_M + L_E\,, \tag{1.1}$$

$$L_M = T - U, \qquad L_E = W_m - W_e\,. \tag{1.2}$$

Here L is the complete Lagrangian, and T, U, W_m, and W_e are, respectively, the kinetic, potential, magnetic, and electric energies of the system. In the quasi-stationary approximation in the system with closed currents the electric energy can be neglected in comparison with the magnetic energy [4] and then

$$L_E = W_m\,. \tag{1.3}$$

If the moving conductor with a current is in an external magnetic field, then its magnetic energy W_m can be written in the form

$$W_m = \frac{1}{2c}\int \mathbf{A}\cdot\mathbf{j}\,dV + \frac{1}{c}\int \mathbf{A}^e\cdot\mathbf{j}\,dV\,. \tag{1.4}$$

Here j is the current density in the conductor, A is the vector potential of the magnetic field of the current j, and A^e is the vector potential of the external magnetic field; integration is performed over the conductor volume. It is obvious that the first term in (1.4) is the energy of the conductor in the self-magnetic field and the second term is its energy in the external field.

It is convenient to introduce a discrete description of the system. We assume that the three-dimensional conductor in question has a countable number of "mechanical" degrees of freedom, corresponding to the generalized coordinates ξ_m and velocities $\dot{\xi}_m$. We represent the current density j in the form of a series in the system $S_i(\mathbf{r})$ of solenoidal vector functions, which is complete with respect to the permissible current distribution functions:

$$\mathbf{j}(\mathbf{r}, t) = \sum_{i=1}^{\infty} q_i^{\cdot}(t)\,\mathbf{S}_i(\mathbf{r})\,. \tag{1.5}$$

The coefficients $q_i^{\bullet}$ of this expansion are the individual "branches" of the current density. Taking $q_i^{\bullet}$ as generalized velocities, we can consider the conductor with a current as a discrete dynamic system with a countable number of "mechanical" (ξ_m and $\dot{\xi}_m$) and "electrical" (q_i and $\dot{q}_i$) coordinates and velocities.

According to the equation

$$\operatorname{rot}\frac{1}{\mu}\operatorname{rot}\mathbf{A} = \frac{4\pi}{c}\mathbf{j}\,, \tag{1.6}$$

the vector potential A is a function of the generalized velocities $\dot{q}$ and geometric coordinates r, $A = A(r, \dot{q})$, and in view of the linearity of (1.6)

$$\mathbf{A}(\mathbf{r}, q^{\cdot}) = \frac{1}{c}\sum_{j=1}^{\infty}\mathbf{A}_j(\mathbf{r})\,q_j^{\cdot}(t)\,. \tag{1.7}$$

We introduce the generalized fluxes Φ_i by the relation

$$\Phi_i(\xi, q^{\cdot}) = \int \mathbf{A}(\mathbf{r}, q^{\cdot})\cdot\mathbf{S}_i(\mathbf{r})\,dV = \frac{1}{c}\sum_{j=1}^{\infty}L_{ij}(\xi)\,q_j^{\cdot}(t). \tag{1.8}$$

By analogy with linear conductors the coefficients

$$L_{ij}(\xi) = \int \mathbf{A}_j(\mathbf{r})\cdot\mathbf{S}_i(\mathbf{r})\,dV$$

can be termed the generalized coefficients of the self-induction and mutual-induction of the corresponding current branches of the current j in expansion (1.5).

Taking (1.5), (1.7), (1.8) into account, we transform the terms of the magnetic energy (1.4)

$$W_{ms} \equiv \frac{1}{2c}\int \mathbf{A}\cdot\mathbf{j}\,dV = \frac{1}{2c^2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}L_{ij}(\xi)\,q_i^{\cdot}(t)\,q_j^{\cdot}(t)\,, \tag{1.9}$$

$$W_{me} \equiv \frac{1}{c}\int \mathbf{A}^e\cdot\mathbf{j}\,dV = \frac{1}{c}\sum_{i=1}^{\infty}q_i^{\cdot}(t)\,\Phi_i^e(t, \xi)\,. \tag{1.10}$$

By analogy with the definition (1.8), in (1.10) we have introduced the generalized external field fluxes

$$\Phi_i^e(t, \xi) = \int \mathbf{A}^e(t, \mathbf{r})\cdot\mathbf{S}_i(\mathbf{r})\,dV\,,$$

in which in place of the dependence on $\dot{q}^e(t)$ the dependence on t appears explicitly.

Summing (1.9) and (1.10), we find the magnetic energy

$$W_m(t, \xi, q^{\cdot}) = \frac{1}{c}\sum_{i=1}^{\infty}q_i^{\cdot}(t)\left[\frac{1}{2c}L_{ij}(\xi)\,q_i^{\cdot}(t) + \Phi_i^e(t, \xi)\right]. \tag{1.11}$$

With account for (1.1), (1.3), (1.11) the complete Lagrangian of the system is written in the form

$$L(t, \xi, \xi^{\cdot}, q^{\cdot}) = L_M(\xi, \xi^{\cdot}) + \frac{1}{c}\sum_{i=1}^{\infty}q_i^{\cdot}\left[\frac{1}{2c}L_{ij}(\xi)\,q_j^{\cdot} + \Phi_i^e(t, \xi)\right]. \tag{1.12}$$

The function L does not depend explicitly on q_i, i.e., the coordinates q_i are cyclic and this means that the generalized impulses corresponding to these coordinates are conserved,

$$p_i = \frac{\partial L}{\partial q_i^{\cdot}} = \text{const.}$$

Therefore it is convenient to describe the system with the aid of the Routh function $R = R(t, \xi, \xi^{\cdot}, p)$, defined by the equality

$$R = L - \sum_{i=1}^{\infty}p_i q_i^{\cdot}\,, \tag{1.13}$$

whose right-hand side must be expressed through the Routh variables t, ξ, $\xi^{\cdot}$, p. For the position coordinates ξ_m the Routh function plays the role of the Lagrange function, while for the cyclic coordinates q_i the function R plays the role of the Hamilton function.

In the case in question the generalized impulse

$$p_i = \frac{1}{c}(\Phi_i + \Phi_i^e) = \text{const} \ , \tag{1.14}$$

i. e., the total magnetic field flux through the current branch contour q_i is conserved:

$$\Phi_{i0} = \Phi_i + \Phi_i^e . \tag{1.15}$$

With the aid of (1.12) and (1.14) the Routh function (1.13) can be written in the form

$$R = L_M - \frac{1}{2c^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} L_{ij} q_i^{\cdot} q_j^{\cdot} \tag{1.16}$$

The last term in (1.16) is the self-magnetic energy (1.9) of the conductor, taken with reversed sign, and therefore

$$R = L_M - W_{ms} \ . \tag{1.17}$$

Thus the dynamics of the electromechanical system in question is described with the aid of the Routh function (1.17), which is the difference between the conventional mechanical Lagrangian and the self-magnetic energy of the system.

If we introduce the generalized potential energy (the Routh potential [5]) $W = U + W_{ms}$, then with account for (1.2) the function R can be written in the form

$$R = T - W \ . \tag{1.18}$$

Let us express R in terms of the Routh variables. To do this we obviously need only transform the magnetic energy W_{ms}.

By virtue of the positive definiteness of the quadratic form (1.9), $\det \| L_{ij} \| \neq 0$, therefore we find from (1.8)

$$q_i^{\cdot} = c \sum_{j=1}^{\infty} L_{ij}^{-1} \Phi_j \ . \tag{1.19}$$

Here $\|L_{ij}^{-1}\|$ is the inverse of the matrix $\|L_{ij}\|$. Using (1.19), we write W_{ms} in the Routh variables

$$W_{ms} = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} L_{ij}^{-1} \Phi_i \Phi_j \ . \tag{1.20}$$

In the simplest case of a linear conductor (1.20) takes the form $W_{ms} = \Phi^2/2L$, where L is the self-induction coefficient, and Φ is the self-magnetic field flux through the conductor aperture; Φ satisfies a relation of the (1.15) type.

2. Now let us examine a slender closed plasma conductor (ring), whose cross-section radius a is small in comparison with the characteristic dimension of the axial line. The plasma is assumed to be inviscid and ideally conducting. The ring is maintained in equilibrium and stabilized relative to smooth disturbances of the zigzag and sausage type by a magnetic field, which in the general case is a combination of constant and quasi-stationary high-frequency fields. Both the surface current flowing through the ring and the magnetic field may have a high-frequency component.

Considering the conductor as an electromechanical system, we describe its motion about the equilibrium position by the Routh function (1.17). We consider the mechanical Lagrangian to be known, thereby limiting ourselves (see Appendix) to finding the self-magnetic energy (1.9). The latter must be expressed through the Routh variables, i. e., in the final analysis through the time t and the geometric coordinates ξ_m describing the disturbance of the ring.

Let us first assume that there is no field in the plasma, and therefore the ring can be considered superconducting. However, to simplify the calculations we consider the current induced in the superconductor by the external magnetic field and the self-magnetic field of the superconductor to be distributed in a thin surface layer, passing to the limit only in the final expressions. In this case the magnetic energy can be written in the form

$$W_{ms}^{ext} = \frac{1}{2c}\int \mathbf{A}\cdot\mathbf{j}\,dV\,. \tag{2.1}$$

Here j is the volume density of the current in the filament, and A is the vector potential of the magnetic field of this current; the integration is performed with respect to the volume of the conductor.

We write the current density j in the form

$$\mathbf{j} = c\,\mathrm{rot}\,\mathbf{M} + \mathbf{j}_I\,, \tag{2.2}$$

where M is a formally introduced magnetization vector which is nonzero only within the conductor. The first term, associated with the magnetization of the medium, yields no contribution to the total current flowing through the cross section of the filament, so that the total current I is determined only by the second term of (2.1)

$$I = \int \mathbf{j}\cdot d\mathbf{f} = \int \mathbf{j}_I\cdot d\mathbf{f}\,.$$

By virtue of the linearity of the field equations

$$\mathbf{A} = \mathbf{A}_M + \mathbf{A}_I\,, \tag{2.3}$$

where A_M and A_I are the vector potentials of the magnetic fields of the magnetization current and the current j_I, respectively. Substituting (2.2) and (2.3) into (2.1), we obtain

$$W_{ms}^{ext} = \frac{1}{2c}\int \mathbf{A}_I\cdot\mathbf{j}_I\,dV + \frac{1}{2}\int \mathbf{A}_M\cdot\mathrm{rot}\,\mathbf{M}\,dV + \frac{1}{2}\int \mathbf{A}_I\cdot\mathrm{rot}\,\mathbf{M}\,dV + \frac{1}{2c}\int \mathbf{A}_M\cdot\mathbf{j}_I\,dV \tag{2.4}$$

The first term can be written in the form

$$\frac{1}{2c}\int \mathbf{A}_I\cdot\mathbf{j}_I\,dV = \frac{\Phi^2}{2L}\,. \tag{2.5}$$

Here Φ is the flux of the magnetic field of the current I through the ring, and L is the self-induction of the ring for the current I. The flux Φ is found from a relation of the (1.15) type,

$$\Phi + \Phi_e = \Phi_0 = \mathrm{const}\,. \tag{2.6}$$

Here Φ_e is the flux of the external field through the contour of the current I, and Φ_0 is the total flux of the field through the ring, which in view of the ideal conductivity is conserved. From the formal viewpoint Φ_0/c is the conserved generalized impulse, corresponding to the cyclic coordinate

$$q = \int I\,dt\,.$$

The second term in (2.4) transforms to the form

$$\frac{1}{2}\int \mathbf{A}_M\cdot\mathrm{rot}\,\mathbf{M}\,dV = \frac{1}{2}\oint(\mathbf{M}\times\mathbf{A}_M)\cdot d\mathbf{S} + \frac{1}{2}\int \mathbf{M}\cdot\mathrm{rot}\,\mathbf{A}_M\,dV\,.$$

The integral over the surface enclosing the conductor and passing everywhere outside it equals zero. Substituting $B_M = \mathrm{rot}\,A_M$ into the volume integral, we obtain

$$\frac{1}{2}\int \mathbf{A}_M\cdot\mathrm{rot}\,\mathbf{M}\,dV = \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_M\,dV \tag{2.7}$$

with integration over the volume of the conductor (there only, $\mathbf{M} \neq 0$). But within the superconductor the magnetic field induction, made up of the external field induction B_e and the induction B_M of the magnetization current field, equals zero, $B = B_e + B_M = 0$. Consequently, within the conductor $B_M = -B_e$.

Thus

$$\frac{1}{2}\int \mathbf{A}_M\cdot\text{rot}\,\mathbf{M}\,dV = -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_e\,dV\,. \tag{2.8}$$

After transformation the third term in (2.4) takes a form analogous to (2.8):

$$\frac{1}{2}\int \mathbf{A}_I\cdot\text{rot}\,\mathbf{M}\,dV = \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_I\,dV \quad (\mathbf{B}_I = \text{rot}\,\mathbf{A}_I)\,. \tag{2.9}$$

Here the integration is again taken over the volume of the conductor. However the field B_I in the conductor, like the current j_I, is concentrated in a thin surface layer and has there a finite magnitude. Therefore, in the limit of an infinitesimally thin layer the integral on the right in (2.9) equals zero and this means that

$$\frac{1}{2}\int \mathbf{A}_I\cdot\text{rot}\,\mathbf{M}\,dV = 0\,. \tag{2.10}$$

After the substitution $j_I = (c/4\pi)$ rot B_I the fourth term in (2.4) can be written in the form

$$\frac{1}{2c}\int \mathbf{A}_M\cdot\mathbf{j}_I\,dV = \int \text{div}\,(\mathbf{B}_I\times\mathbf{A}_M)\,dV + \frac{1}{8\pi}\int \mathbf{B}_I\cdot\text{rot}\,\mathbf{A}_M\,dV.$$

The integral of div $(B_I \times A_M)$ transforms into an integral over an infinitely distant surface and vanishes. Substituting B_M = rot A_M into the second integral, we find that

$$\frac{1}{2c}\int \mathbf{A}_M\cdot\mathbf{j}_I\,dV = \frac{1}{8\pi}\int \mathbf{B}_I\cdot\mathbf{B}_M\,dV\,. \tag{2.11}$$

The integration in (2.11) must be made over the entire space. However, as we noted previously, the field B_I in the conductor is nonzero only in a thin surface layer; therefore in actuality the integration can be carried out over only the space outside the conductor.

Since the filament is thin and its perturbations are smooth over segments which are small in comparison with the perturbation wavelength, it can be considered to be a cylinder and then we can use for M, B_I, and B_M the expressions which hold in the case of a cylindrical conductor. We also assume that the external field changes very little over a distance of the order of the filament radius a.

The field B_M outside the superconducting cylinder in a uniform external field has the form [6] $\mathbf{B}_M = (a/r)^2\,[\mathbf{B}_{e\perp} - 2\mathbf{n}\,(\mathbf{n}\cdot\mathbf{B}_{e\perp})]$. Here n is the unit normal to the cylinder surface, and $\mathbf{B}_{e\perp}$ is the external field component perpendicular to the centerline of the conductor. On the other hand, $\mathbf{B}_I = B_a\,(a/r)\,(\boldsymbol{\tau}\times\mathbf{n})$, where $B_a = 2I/ca$, and τ is the unit vector of the tangent to the centerline of the filament. It follows from these formulas that

$$\mathbf{B}_I\cdot\mathbf{B}_M = -(a/r)^3 B_a B_{e\perp}\sin\theta\,, \tag{2.12}$$

where θ is the angle in the plane of the filament cross section measured from the vector $\mathbf{B}_{e\perp}$. Integration of (2.12) with respect to θ from 0 to 2π in the volume integral (2.11) yields zero, since in the "locally cylindrical" approximation

$$\frac{1}{2c}\int \mathbf{A}_M\cdot\mathbf{j}_I\,dV = 0\,. \tag{2.13}$$

In view of (2.10) and (2.13) the self-magnetic energy W_{ms}^{ext} of the superconducting ring in the locally cylindrical approximation is determined only by the first two terms of (2.4). Summing (2.5) and (2.8) with account for (2.6), we find that

$$W_{ms}^{ext} = \frac{(\Phi_0-\Phi_e)^2}{2L} - \frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}_e\,dV\,. \tag{2.14}$$

The self-induction coefficient L as a function of the geometric coordinates ξ_m can be found from the formula [6]

$$L = \iint\limits_{R>a/2} \frac{d\mathbf{l}_1\cdot d\mathbf{l}_2}{R}\,, \tag{2.15}$$

where dl_1, dl_2 are arc elements of the disturbed conductor centerline, and R is the distance between them. In particular, for a disturbed circular ring the coefficient L was obtained in [1]. The flux Φ_e is expressed through ξ_m and the given external field:

$$\Phi_e = \int \mathbf{A}^e \cdot d\mathbf{l} . \tag{2.16}$$

As for the magnetization vector **M**, in the approximation being considered it is found from the known formula for the magnetization of a cylinder in a uniform external field [6]

$$\mathbf{M} = -\mathbf{B}_e / 4\pi (1-n) , \tag{2.17}$$

where n = 0 for the longitudinal field and n = 1/2 for the transverse field.

If there is now a frozen-in field B_i in the plasma, its energy

$$W_{ms}^{int} = \frac{1}{8\pi} \int \mathbf{B}_i^2 \, dV ,$$

which together with W_{ms}^{ext} (2.14) constitutes the complete self-magnetic energy of the plasma ring. Let the field B_i at the initial time be directed along the filament centerline, $\mathbf{B}_i = B_i \boldsymbol{\tau}$, and let it nearly uniform in each cross section. Then as a result of the ideal conductivity it will remain nearly uniform and collinear with the centerline during the perturbation as well, i.e.,

$$\int \mathbf{B}_i \cdot d\mathbf{S} = \langle B \rangle_i S = \Phi_i = \text{const} ,$$

where $\langle B \rangle_i$ is the average value of the field B_i across the section of the filament, and S is the section area. Since the filament is thin, we can assume that $B_i = \langle B \rangle_i$. Then

$$W_{ms}^{int} = \frac{\Phi_i^2}{8\pi} \int \frac{dl}{S} ,$$

with integration along the filament centerline.

The total self-magnetic field energy of the ring is given by the expression

$$W_{ms} = \frac{(\Phi_0 - \Phi_e)^2}{2L} - \frac{1}{2} \int \mathbf{M} \cdot \mathbf{B}_e \, dV + \frac{\Phi_i^2}{8\pi} \int \frac{dl}{S} , \tag{2.18}$$

where L, Φ_e, and **M** are found from (2.15)–(1.17). We emphasize that W_{ms} is represented, although not explicitly, in the Routh variables; since each term in (2.18) is expressed in terms of the given magnetic field, which in the general case depends on the time and the geometric coordinates.

Thus, if the mechanical Lagrangian is known, by determining from the given external field and the filament perturbation the self-magnetic energy we can find the Routh function describing the filament motion about the equilibrium position.

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